PMM U.S.S.R., Vol.54, No.1, pp. 83-87, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

WAVE PROPAGATION IN AN ELASTIC LONGITUDINALLY INHOMOGENEOUS CYLINDER*

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The problem of the propagation of stationary waves in an elastic waveguide that is inhomogeneous along its length is examined. A mathematical foundation is given for the algorithm for solving the problem to determine the wave fields in piecewise-homogeneous waveguides. Results of computations of the wave reflection and transmission coefficients in a strip with two interface boundaries of the material properties are presented. A relation is obtained between the extremal properties of these coefficients and the different resonance phenomena originating in a longitudinally inhomogeneous waveguide.

1. We consider the problem of the propagation of stationary waves proportional to exp $(-i\omega t)$ in an inhomogeneous elastic anisotropic cylinder $S\{|x_1|<\infty;\, x_2, x_3\in D\}$ consisting of subdomains $S_0=(-\infty, \xi_0]\times D$, $S_k=[\xi_{k-1}, \xi_k]\times D$ $(k=1, 2, \ldots, n-1)$, $S_n=[\xi_{n-1}, \infty)\times D$ ($D\in R^2$ is the transverse cross-section of the cylinder that is a bounded domain with a smooth boundary in the plane x_2, x_3). Let Γ_k denote the cylindrical part of the boundary of the subdomain S_k. In the general case the subdomains S_k differ in the nature of the boundary conditions on Γ_k, the elastic moduli c_{ijkl}, the density ρ.

We introduce the notation $\mathbf{u}=\{u_n\}_{n=1}^3$, $\boldsymbol{\sigma}_m=\{\sigma_{mn}\}_{n=1}^3$, $\boldsymbol{\varepsilon}_m(\mathbf{u})=\{\varepsilon_{mn}\}_{n=1}^3=\{{}^1/_2(\partial_m u_n+\partial_n u_m)\}_{n=1}^3$, where u_n, σ_{mn}, ε_{mn} are the displacement, stress, and strain amplitudes, and $\partial_n=\partial/\partial x_n$. Taking Hooke's law into account the stress vector $\boldsymbol{\sigma}_m$ can be represented for an anisotropic material as follows (A_m, B_m are (3 x 3) matrix operators):

$$\boldsymbol{\sigma}_m(\mathbf{u})=A_m\partial\mathbf{u}/\partial x_1-iB_m\mathbf{u} \tag{1.1}$$
$$A_m=\|A_m(k,l)\|=\|c_{mkl1}\|,\quad B_m=\|B_m(k,l)\|=i\|c_{mkl2}\partial_2+c_{mkl3}\partial_3\|,\quad k, l=1,2,3;\quad i^2=-1$$

Suppose $H=H(D)$ is a Hilbert space of three-component vector-functions, square integrable in D and $H_\alpha=H_\alpha(D)$ is the Sobolev-Slobodetskii space /1/. We introduce the six-component vector $\mathbf{W}=\{\mathbf{u}, \boldsymbol{\sigma}_1\}$ and the Hilbert spaces of six-component vector-functions $H'=H\oplus H$ and $H_{\alpha\beta}'=H_\alpha\oplus H_\beta$ into the considerations. The mean power flux P per period $T=2\pi/\omega$ through a section $x=x_1=\text{const}$ can then be represented in the form /2/

$$P(\mathbf{W})=\frac{\omega}{4}(J\mathbf{W}, \mathbf{W})_{H'}=\frac{\omega}{4}[\mathbf{W}, \mathbf{W}],\quad J=i\begin{Vmatrix}O & -I\\ I & O\end{Vmatrix} \tag{1.2}$$

(O is the zero, and I the identity operator in H).

Let $H_\varepsilon(S_k)$ be an energetic space of vector-functions with the norm

$$\|\mathbf{u}\|^2=\int_{\xi_{k-1}}^{\xi_k}\Pi(\mathbf{u}, \mathbf{u})\,dx,\quad \Pi(\mathbf{u}, \mathbf{u})'=\sum_{m=1}^3(\boldsymbol{\sigma}_m(\mathbf{u}), \boldsymbol{\varepsilon}_m(\mathbf{u}))_H \tag{1.3}$$

$C(H_\alpha)$, $C(H_{\alpha\beta}')$ are spaces of vector-functions $\mathbf{u}(x)$ and $\mathbf{W}(x)$ continues in $x\in R$ with values in H_α, $H_{\alpha\beta}'$ respectively.

2. The equations of stationary vibrations

$$\partial_m\boldsymbol{\sigma}_m+\rho\omega^2\mathbf{u}=0 \tag{2.1}$$

in each subdomain S_k can be represented in terms of the vector $\mathbf{W}$ taking the relationship (1.1) into account in the following form (T^k is a matrix differential (6 x 6) form)

*Prikl. Matem. Mekhan., 54,1,103-108,1990

$$\left(i\frac{\partial}{\partial x}+|T^k\right)\mathbf{W}^k=0,\quad T^k=\begin{Vmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{Vmatrix} \tag{2.2}$$

$$t_{11}=A_1^{-1}B_1,\quad t_{21}=i\,(B_1{}^*A_1^{-1}B_1-V+\rho\omega^2I)$$
$$t_{12}=-iA_1^{-1},\quad t_{22}=B_1{}^*A_1^{-1},\quad V=i\,(\partial_2B_2+\partial_3B_3)$$

To be specific, one of the following conditions

$$\mathbf{u}\,|_{\Gamma_k}=0 \tag{2.3}$$

or

$$\sigma_n\,|_{\Gamma_k}=(\sigma_2n_2+\sigma_3n_3)\,|_{\Gamma_k}=0 \tag{2.4}$$

is given on the side surfaces Γ_k (n is a vector normal to the side surface).

The continuity conditions

$$\mathbf{W}^k(\xi_k)=\mathbf{W}^{k+1}(\xi_k) \tag{2.5}$$

are satisfied in the sections $x=\xi_k$.

A normal wave incident on the boundary $x=\xi_0=0$ from $-\infty$ and determined by the elementary solutions $\mathbf{W}_s^0(x)=\mathbf{W}_*(x)$ (see (3.2)) for which $P(\mathbf{W}_*)>0$ is the source of the vibrations.

3. The matrix differential form T^k with the boundary conditions (2.3) or (2.4) will generate the operators T_0^k and T_1^k, respectively. The linear bundles

$$(T_\beta{}^k-\gamma^k)\,\mathbf{V}^k=0\quad(\beta=0,1) \tag{3.1}$$

equivalent to quadratic bundles investigated in /3/ correspond to the problems (2.2) and (2.3) and (2.2) and (2.4). Later the superscript k, indicating belonging to the subdomain S_k, will be omitted where it will not result in confusion.

Let γ_t be the eigenvalue of the operator T_β, $\mathbf{V}_{0t},\mathbf{V}_{1t},\ldots,\mathbf{V}_{pt}$ are the corresponding eigen- and associated vector-functions (Jordan chain). A vector-function of the form

$$\mathbf{W}_t(x)=\exp\,(i\gamma_tx)\left[\frac{(ix)^p}{p!}\mathbf{V}_{0t}+\frac{(ix)^{p-1}}{(p-1)!}\mathbf{V}_{1t}+\ldots+\mathbf{V}_{pt}\right] \tag{3.2}$$

is called an elementary solution of (2.2) satisfying the conditions (2.3) or (2.4).

Relying on the results of /2-5/, the following properties of the operator T_β, their spectra $\sigma(T_\beta)$ and systems of elementary solutions can be formulated.

1°. The operator T_β is J-selfadjoint in H', i.e., $(JT_\beta)^*=JT_\beta$.

2°. The spectrum $\sigma(T_\beta)=\Lambda^+\cup\Lambda^-\cup\Lambda^\circ$, where $\gamma_t{}^+\in\Lambda^+$ if $\operatorname{Im}\gamma_t{}^+>0$, $\gamma_t{}^-\in\Lambda^-$, if $\operatorname{Im}\gamma_t{}^-<0$, $\gamma_s{}^\circ\in\Lambda^\circ$, if $\operatorname{Im}\gamma_s{}^\circ=0$. The sets $\Lambda^\pm$ have limit points at infinity ($t=\pm1,\pm2,\ldots$), the set Λ° is finite ($s=\pm1,\pm2,\ldots,\pm N$). The eigenvalues $\gamma_t{}^\pm$, $\gamma_s{}^\circ$ satisfy the following symmetry properties: $\gamma_t{}^+=\gamma_t$, $\gamma_t{}^-=\gamma_t{}^*$, $\gamma_{-t}{}^+=-\gamma_t{}^*$, $\gamma_{-t}{}^-=-\gamma_t$, $\gamma_{-s}{}^\circ=-\gamma_s{}^\circ$. Complex-conjugate quantities are marked with an asterisk.

3°. A critical frequency $\omega_0>0$ exists for the operator T_0 such that $\Lambda^\circ=\{0\}$ for $\omega<\omega_0$.

4°. The system of Jordan chains $\{\mathbf{V}_{lt}\}$ is complete in the space $H'_{\alpha,\alpha-1/2}$, $0\leqslant\alpha\leqslant 1/2$.

Let $M=\{\mathbf{W}_t(x)\}$ denote a system of elementary solutions and in conformity with the property 2° let $M=M^+\cup M^-\cup M^\circ$.

5°. A partition $M^\circ=M_0{}^+\cup M_0{}^-$, exists such that $P(\mathbf{W}_s{}^+)>0$ for any $\mathbf{W}_s{}^+(x)\in M_0{}^+$ while $P(\mathbf{W}_s{}^-)<0$ for any $\mathbf{W}_s{}^-(x)\in M_0{}^-$. The elementary solution $\mathbf{W}_s{}^+(x)\in M_0{}^+$ transfers energy from left to right while the elementary solution $\mathbf{W}_t{}^+(x)\in M^+$ damps out exponentially from left to right. The elementary solutions $\mathbf{W}_s{}^-(x)\in M_0{}^-$, $\mathbf{W}_t{}^-(x)\in M^-$ possess opposite properties.

6°. The following orthogonality conditions hold for elements of the sets $M^\pm$, $M_0{}^\pm$:

$$[\mathbf{W}_s{}^\pm,\mathbf{W}_l{}^\pm](x)=\pm\delta_{sl},\ [\mathbf{W}_s{}^\pm,\mathbf{W}_l{}^\mp](x)=0,\ \mathbf{W}_s{}^\pm,\mathbf{W}_l{}^\pm\in M_0{}^\pm \tag{3.3}$$
$$[\mathbf{W}_t{}^\pm,\mathbf{W}_r{}^\pm](x)=0,\ [\mathbf{W}_t{}^\pm,\mathbf{W}_r{}^\mp](x)=\exp\,(\pm i\theta_r)\,\delta_{tr}$$
$$\mathbf{W}_t{}^\pm,\mathbf{W}_r{}^\pm\in M^\pm$$
$$[\mathbf{W}_s,\mathbf{W}_t](x)=0,\ \mathbf{W}_s\in M_0{}^\pm,\ \mathbf{W}_t\in M^\pm$$

We define the space of homogeneous generalized solutions $H_0(S_k)$ in each subdomain S_k ($k=1,2,\ldots,n-1$). When the boundary condition (2.3) is given on Γ_k, we will say that the vector-function $\mathbf{u}^k(x)\in H_0(S_k)$, if $\mathbf{u}(x)$ satisfies condition (2.3) and the integral identity

$$\int\limits_{\xi_{k-1}}^{\xi_k}[\Pi(\mathbf{u}^k,\boldsymbol{\varphi}^k)-\omega^2(\rho^k\mathbf{u}^k,\boldsymbol{\varphi}^k)_H]\,d\xi=0 \tag{3.4}$$

for an arbitrary smooth vector-function $\varphi^k(x)$ finite in the segment $[\xi_{k-1}, \xi_k]$.

In the case of the boundary conditions (2.4), it is sufficient to require satisfaction of just the integral identity (3.4) to define the space of homogeneous functions.

We will assume for the subdomains S_0, S_n that the identity (3.4) is satisfied even for the vector-function satisfying the condition

$$\Pi(\mathbf{u}^\alpha, \mathbf{u}^\alpha) \leqslant c(1+|x|^l), \quad (\alpha = 0, n), \quad l \geqslant 2 \max_{\alpha, s} p_s^\alpha$$

(p_s^α are the lengths of the Jordan chains).

The space $H_0'(S_k)$ of homogeneous solutions of Eq.(2.2) is defined as the set of vector-functions $\mathbf{W}^k(x) = \{\mathbf{u}^k(x), \sigma_1(\mathbf{u}^k)\}$, where $\mathbf{u}^k(x) \in H_0(S_k)$, and $\sigma_1(\ldots)$ is an operator acting from $H_{1/2}(D)$ into $H_{-1/2}(D)$.

We obtain the following theorem from the properties 4°-6°.

Theorem 1. Let the vector-function be $\mathbf{u}^k \in H_0(S_k)$, $k = 0, 1, \ldots, n$. Then: 1) the continuity condition (2.5) should be understood in the sense of the metric of the space $C'(H'_{1/2,-1/2})$; 2) the following representation hold

$$\mathbf{W}^k(x) = \sum_s {}_1 C_s^{k+} \mathbf{W}_s^{k+}(x-\xi_{k-1}) + \sum_s {}_2 C_s^{k-} \mathbf{W}_s^{k-}(x-\xi_k) + \tag{3.5}$$
$$\sum_t {}_3 C_t^{k+} \mathbf{W}_t^{k+}(x-\xi_{k-1}) + \sum_t {}_4 C_t^{k-} \mathbf{W}_t^{k-}(x-\xi_k) \quad (k = 1, 2, \ldots, n-1)$$

Here $\Sigma_{1,2,3,4}$ denotes summation over all elementary solutions from M_0^{k+}, M_0^{k-}, M^{k+}, M^{k-}, respectively, and $C_s^{k\pm}$, $C_t^{k\pm}$ are constants. In the case $k = 0$ we should set $\Sigma_3 = 0$, in (3.5) and $\Sigma_4 = 0$ in the case when $k = n$.

Remark 1. The constants $C_s^{\circ+}$, C_s^{n-} differ from zero if there are vibrations sources as $x \to -\infty$ and $x \to \infty$, respectively.

Remark 2. If the vector $\mathbf{u}^k(x)$ is already determined in some manner, then the constants $C_r^{k\pm}$ can be found from the relationships

$$\begin{aligned} &C_s^{k+} = [\mathbf{W}_0^k, \mathbf{W}_s^{k+}(0)], \quad C_s^{k-} = [\mathbf{W}_0^k, \mathbf{W}_s^{k-}(-l_k)], \quad \mathbf{W}_s^{k\pm} \in M_0^{k\pm} \\ &C_t^{k+} = \exp(-i\theta_t)[\mathbf{W}_0^k, \mathbf{W}_t^{k-}(0)] \\ &C_t^{k-} = \exp(i\theta_t)[\mathbf{W}_0^k, \mathbf{W}_t^{k+}(-l_k)], \quad \mathbf{W}_t^{k\pm} \in M^{k\pm} \\ &\mathbf{W}_0^k = \{\mathbf{u}^k(\xi_{k-1}), \sigma_1^k(\xi_{k-1})\}, \quad l_k = \xi_k - \xi_{k-1} \end{aligned} \tag{3.6}$$

These representations extend the results in /6/, related to over-expansion of the solutions obtained by the superposition method in series of homogeneous solutions.

Let us define the vector-functions $\mathbf{u}(x) = \{\mathbf{u}^k(x), x \in [\xi_{k-1}, \xi_k]\}$, $\mathbf{W}(x) = \{\mathbf{W}^k(x), x \in [\xi_{k-1}, \xi_k]\}$ on the whole axis $x \in R$. We will say that $\mathbf{u}(x) \in H_0(S)$, $\mathbf{W}(x) \in H_0'(S)$, if $\mathbf{u}^k(x) \in H_0(S_k)$, $\mathbf{W}^k(x) \in H_0'(S_k)$.

4. In conformity with the problem formulated $\mathbf{W}^\circ(x) = \mathbf{W}_* + \mathbf{W}^{\circ-}$ and the radiation conditions have the following form

$$P(\mathbf{W}^{\circ-}) \leqslant 0, \quad P(\mathbf{W}^n) \geqslant 0 \tag{4.1}$$

Bearing in mind the possibility of using different approximate methods to solve problem (2.2)-(2.5), we will present several definitions of its generalized solution.

Definition 1. We call the vector-function $\mathbf{u} \in H_0(S)$ satisfying the continuity condition

$$\mathbf{u}^k(\xi_k) = \mathbf{u}^{k+1}(\xi_k) \tag{4.2}$$

in the metric $C(H_{1/2})$, the radiation conditions (4.1), and the integral identity

$$\Psi_1(\mathbf{u}, \varphi) + l_1(\varphi) = 0, \quad \forall \varphi \in H_0(S) \tag{4.3}$$

the generalized solution. Here

$$\Psi_1(\mathbf{u}, \varphi) = \sum_{k=0}^{n} [(\sigma_1^k - \sigma_1^{k+1}, \varphi^k)_H(\xi_k) - (\sigma_1^k - \sigma_1^{k-1}, \varphi^k)_H(\xi_{k-1})] \tag{4.4}$$
$$l_1(\varphi) = (\sigma_*, \varphi^0 + \varphi^1)_H(\xi_0), \quad \sigma_* = \sigma_1(\mathbf{u}_*)$$

Definition 2. We call the vector-function $\mathbf{u} \in H_0(S)$ satisfying the continuity condition

$$\sigma_1^k(\xi_k) = \sigma_1^{k+1}(\xi_k) \tag{4.5}$$

in the metric $C(H_{-1/2})$, the radiation conditions (4.1), and the integral identity

$$\Psi_2(\mathbf{u}, \boldsymbol{\varphi}) + l_2(\boldsymbol{\varphi}) = 0, \quad \forall \boldsymbol{\varphi} \in H_0(S) \tag{4.6}$$

the generalized solution. Here

$$\Psi_2(\mathbf{u}, \boldsymbol{\varphi}) = \sum_{k=0}^{n} [(\mathbf{u}^k - \mathbf{u}^{k+1}, \boldsymbol{\sigma}_1(\boldsymbol{\varphi}^k))_H(\xi_k) - (\mathbf{u}^k - \mathbf{u}^{k-1}, \boldsymbol{\sigma}_1(\boldsymbol{\varphi}^k))_H(\xi_{k-1})] \tag{4.7}$$
$$l_2(\boldsymbol{\varphi}) = (\mathbf{u}_*, \boldsymbol{\sigma}_1(\boldsymbol{\varphi}^0) + \boldsymbol{\sigma}_1(\boldsymbol{\varphi}^1))_H(\xi_0)$$

Definitions 1 and 2 are based on the use of the known Hamilton-Lagrange and Castigliano variational principles, respectively. Either the continuity condition (4.2), or (4.5) figures in each of the definitions presented. Below we formulate a definition of a generalized solution in which the continuity conditions are natural, i.e., result from the variational principle. To do this we introduce the functionals

$$\Psi(\mathbf{u}, \boldsymbol{\varphi}) = i[\Psi_2(\mathbf{u}, \boldsymbol{\varphi}) - \Psi_1(\mathbf{u}, \boldsymbol{\varphi})] \tag{4.8}$$
$$l(\boldsymbol{\varphi}) = i[l_2(\boldsymbol{\varphi}) - l_1(\boldsymbol{\varphi})]$$

Here $\mathbf{u}, \boldsymbol{\varphi} \in H_0(S)$.

Setting $\mathbf{W} = \{\mathbf{u}, \boldsymbol{\sigma}_1(\mathbf{u})\}$, $\boldsymbol{\eta} = \{\boldsymbol{\varphi}, \boldsymbol{\sigma}_1(\boldsymbol{\varphi})\}$, and using the relationship $[\mathbf{W}^k, \boldsymbol{\eta}^k](x) = \text{const}$, resulting from the law of conservation of energy, the functionals (4.8) can be given the form

$$\Psi(\mathbf{u}, \boldsymbol{\varphi}) = \Phi(\mathbf{W}, \boldsymbol{\eta}) = [\mathbf{W}^\circ - \mathbf{W}^1, \boldsymbol{\eta}^\circ](\xi_0) + \sum_{k=1}^{n-1} \{[\mathbf{W}^{k-1}, \boldsymbol{\eta}^k](\xi_{k-1}) - \tag{4.9}$$
$$[\mathbf{W}^{k+1}, \boldsymbol{\eta}^k](\xi_k)\} + [\mathbf{W}^{n-1} - \mathbf{W}^n, \boldsymbol{\eta}^n](\xi_{n-1}), \; l(\boldsymbol{\varphi}) = m(\boldsymbol{\eta}) = [\mathbf{W}_*, \boldsymbol{\eta}^\circ + \boldsymbol{\eta}^1](\xi_0)$$

Definition 3. We call the vector-function $\mathbf{W} \in H_0'(S)$ satisfying the radiation conditions (4.1) and the integral identity

$$\Phi(\mathbf{W}, \boldsymbol{\eta}) + m(\boldsymbol{\eta}) = 0, \quad \forall \boldsymbol{\eta} \in H_0'(S) \tag{4.10}$$

the generalized solution of the problem.

On the basis of representations (3.5), the initial boundary-value problem can be reduced to a system of linear algebraic equations in the constants $C_r^{k\pm}$. To do this it is sufficient to substitute (3.4) into (4.10) and then successively to set $\boldsymbol{\eta}^k = \mathbf{W}_r^{k+}, \mathbf{W}_r^{k-}$ $(k = 0, 1, \ldots, n-1, n)$.

From relationships (1.2), (3.3) and (3.5) we obtain the following theorem.

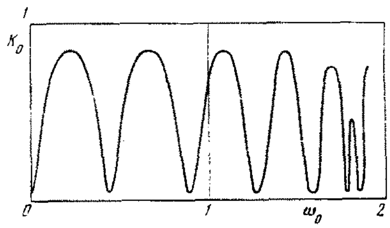
Theorem 2. The mean power flux per period $P(\mathbf{W}^k)$ through a transverse section $x = \text{const}$ of the subdomain S_k is determined by the relationship

$$P(\mathbf{W}^\circ) = P(\mathbf{W}_*) + P(\mathbf{W}^{\circ-}), \; P(\mathbf{W}^{\circ-}) = -\frac{\omega}{4}\sum_{s=1}^{N_0} |C_s^{\circ-}|^2 \tag{4.11}$$
$$P(\mathbf{W}^k) = \frac{\omega}{4}\left\{\sum_{s=1}^{N_k}(|C_s^{k+}|^2 - |C_s^{k-}|^2) + 2\operatorname{Re}\sum_t {}_3\exp(i\gamma_t^k l_k)\exp(i\theta_t^k)C_t^{k+}C_t^{k-*}\right\}$$
$$(k = 1, 2, \ldots, n-1)$$
$$P(\mathbf{W}^n) = \frac{\omega}{4}\sum_{s=1}^{N_n}|C_s^{n+}|^2$$

We set $P(\mathbf{W}_*) = 1$, and we call the quantity $K_0 = -P(\mathbf{W}^{\circ-})$ the reflection coefficient, and $K_n = P(\mathbf{W}^n)$ the transmission coefficient, where $K_0 + K_n = 1$.

In a number of cases expressions (4.11) enable certain general deductions to be drawn regarding the properties of inhomogenous waveguides and controlling them. For instance, let at least one subdomain S_q exist in which there are no homogeneous waves (see property 3°). In this case, $C_s^{q+} = C_s^{q-} = 0$ in the appropriate expression in (4.11) and energy transfer occurs because of interaction of a pair of inhomogeneous waves $\mathbf{W}_t^{q+}(x - \xi_{q-1})$, $\mathbf{W}_t^{q-}(x - \xi_q)$. It is seen here that as $l_q = \xi_q - \xi_{q-1}$ increases, the flux decreases exponentially and asymptotically $P(\mathbf{W}^q) = O[\exp(-l_q \operatorname{Im}\gamma_1^{q+})]$.

The theory was used to investigate the wave fields in a planar inhomogeneous isotropic waveguide $(D = [-h, h], x_3 \equiv 0)$ with two material properties interfacial boundaries, i.e., for $n = 2$ and the boundary conditions (2.4) (the case $n = 1$ is investigated in /7/). It was assumed in the computations that $\lambda_0 = \lambda_2 = 2.06$, $\mu_0 = \mu_2 = 1.53$, $\lambda_1 = 0.59$, $\mu_1 = 0.26$ $(\times 10^{11} H/m^2)$, $\rho_0 = \rho_2 = 1.87$, $\rho_1 = 0.27$ $(\times 10^4$ kg/ m$^3)$, $l_1 = \xi_1 - \xi_0 = 5h$.



The behaviour of the reflection coefficient K_0 as a function of the dimensionless frequency $\omega_0 = \omega h (\rho_0/\mu_0)^{1/2}$ is shown in the figure for incidence of the first normal compression-tension wave.

The sharp minima $K(\omega_0)$ correspond to an intense increase in the vibrations amplitudes in the rectangle S_1. The appropriate frequencies can be considered to be resonant for S_1. Unlike the results obtained by applied one-dimensional theories, taking account of dispersion in the waveguide results in compression of the points of minimum K_0. The reflection coefficient also has a minimum at the boundary resonance frequency ω_* /7/, which depends slightly on the change in the length of the insert l_1 as calculations have shown.

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Translated by M.D.F.